13 INDETERMINATE STRUCTURES

Nomenclature

A	area	ft^2
C	carry-over factor	–
COM	carry-over moment	ft–lbf
D	distribution factor	–
E	modulus of elasticity	lbf/ft^2
F	fixity factor	–
I	moment of inertia	in^4
k	spring constant	ft–lbf/radian
K	stiffness factor	ft–lbf
L	length	ft
M	moment	ft-lbf
P	load	lbf
R	relative stiffness, or reaction	ft–lbf, or lbf
S	force in truss member	lbf
T	temperature	°F
u	dummy force in truss member	lbf
w	continuous load	lbf/ft
x*	distance to centroid of moment diagram	ft

Symbols

δ	deflection	ft
σ	stress	lbf/ft^2
α	coefficient of thermal expansion	1/°F
θ	angle	radians
Δ	deflection	ft

Subscripts

BM	bending moment
c	concrete
M	due to moment M
P	due to force P
s	steel

1 INTRODUCTION

A structure that is statically indeterminate is one for which the equations of statics are not sufficient to determine all reactions, moments, and internal stress distributions. Additional formulas involving deflection are required to completely determine these unknowns. Although there are a large number of problem types which are statically indeterminate, this chapter is primarily concerned with the following cases:

- Beams on more than two supports
- Trusses with more than 2(# joints)–3 members
- Rigid frames

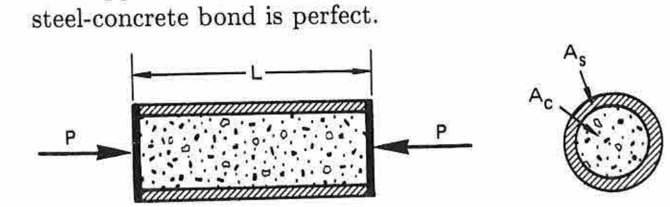
The *degree of redundancy* is equal to the number of reactions or members that would have to be removed in order to make the structure statically determinate. For example, a two-span beam on three simple supports is redundant to the first degree.

2 CONSISTENT DEFORMATION

The method of *consistent deformation* can be used to evaluate simple structures consisting of two or three members in tension or compression. This method is simple to learn and apply. The method makes use of geometry to develop relationships between the deflections (deformations) between different members or locations on the structure.

Example 13.1

A pile is constructed of concrete with a steel jacket. What are the stresses in the steel and concrete if a load P is applied? Assume the end caps are rigid and the steel-concrete bond is perfect.



Let P_c and P_s be the loads carried by the concrete and steel respectively. Then,

$$P_c + P_s = P \qquad 13.1$$

The deformation of the steel is

$$\delta_s = \frac{P_s L}{A_s E_s} \qquad 13.2$$

Similarly, the deflection of the concrete is

$$\delta_c = \frac{P_c L}{A_c E_c} \qquad 13.3$$

But, $\delta_c = \delta_s$ since the bonding is perfect. Therefore,

$$\frac{P_c L}{A_c E_c} - \frac{P_s L}{A_s E_s} = 0 \qquad 13.4$$

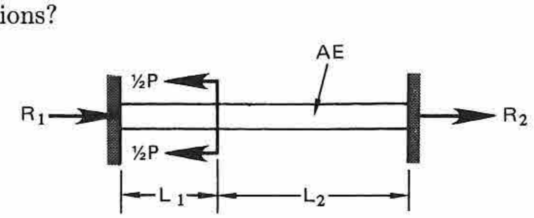
Equations 13.1 and 13.4 are solved simultaneously to determine P_c and P_s. The respective stresses are:

$$\sigma_s = \frac{P_s}{A_s} \qquad 13.5$$

$$\sigma_c = \frac{P_c}{A_c} \qquad 13.6$$

Example 13.2

A uniform bar is clamped at both ends and the axial load applied near one of the supports. What are the reactions?



The first required equation is

$$R_1 + R_2 = P \qquad 13.7$$

The shortening of section 1 due to the reaction R_1 is

$$\delta_1 = \frac{-R_1 L_1}{AE} \qquad 13.8$$

The elongation of section 2 due to the reaction R_2 is

$$\delta_2 = \frac{R_2 L_2}{AE} \qquad 13.9$$

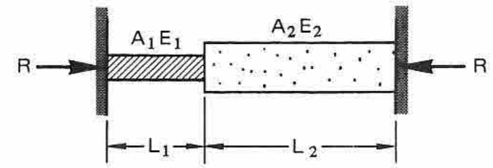
However, the bar is continuous, so $\delta_1 = -\delta_2$. Therefore,

$$R_1 L_1 = R_2 L_2 \qquad 13.10$$

Equations 13.7 and 13.10 are solved simultaneously to find R_1 and R_2.

Example 13.3

The non-uniform bar shown is clamped at both ends. What are the reactions if a temperature change of ΔT is experienced?



The thermal deformations of sections 1 and 2 can be calculated directly.

$$\delta_1 = \alpha_1 L_1 \Delta T \qquad 13.11$$

$$\delta_2 = \alpha_2 L_2 \Delta T \qquad 13.12$$

The total deformation is $\delta = \delta_1 + \delta_2$. However, the deformation can also be calculated from the principles of mechanics.

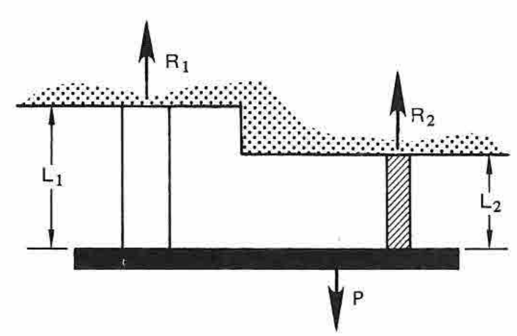
$$\delta = \frac{RL_1}{A_1 E_1} + \frac{RL_2}{A_2 E_2} \qquad 13.13$$

Combining equations 13.11 through 13.13 produces an equation that can be solved directly for R.

$$(\alpha_1 L_1 + \alpha_2 L_2)\Delta T = \left(\frac{L_1}{A_1 E_1} + \frac{L_2}{A_2 E_2}\right) R \qquad 13.14$$

Example 13.4

The beam shown is supported by dissimilar members. What are the forces in the members? Assume the bar is rigid and remains horizontal.[1]



The required equilibrium condition is

$$R_1 + R_2 = P \qquad 13.15$$

The elongations of the two tension members are

$$\delta_1 = \frac{R_1 L_1}{A_1 E_1} \qquad 13.16$$

$$\delta_2 = \frac{R_2 L_2}{A_2 E_2} \qquad 13.17$$

[1] Notice that the location of force P is not specified.

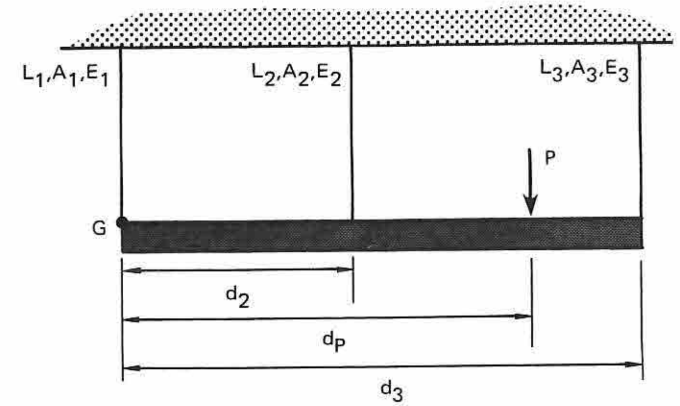
If the horizontal bar remains horizontal, then $\delta_1 = \delta_2$.

$$\frac{R_1 L_1}{A_1 E_1} = \frac{R_2 L_2}{A_2 E_2} \qquad 13.18$$

Equations 13.15 and 13.18 are solved simultaneously to find R_1 and R_2.

Example 13.5

The beam shown is supported by dissimilar members. The bar is rigid, but is not constrained to remain horizontal. What are the reactions in the vertical members?



The forces in the supports are R_1, R_2, and R_3. Any of these may be tensile (positive) or compressive (negative).

$$R_1 + R_2 + R_3 = P \qquad 13.19$$

The changes in lengths are

$$\delta_1 = \frac{R_1 L_1}{A_1 E_1} \qquad 13.20$$

$$\delta_2 = \frac{R_2 L_2}{A_2 E_2} \qquad 13.21$$

$$\delta_3 = \frac{R_3 L_3}{A_3 E_3} \qquad 13.22$$

Since the bar is rigid, the deflections will be proportional to the distance from point G.

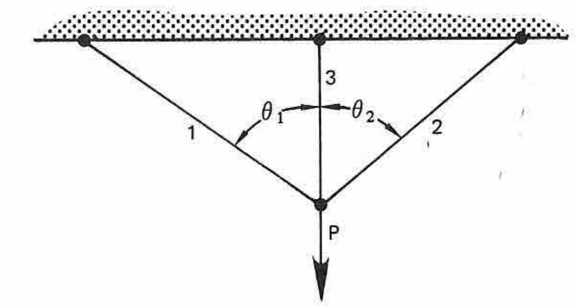
$$\delta_2 = \delta_1 + \frac{d_2}{d_3}(\delta_3 - \delta_1) \qquad 13.23$$

Moments can be summed about point G to give a third equation.

$$M_G = R_3 d_3 + R_2 d_2 - P d_P = 0 \qquad 13.24$$

Example 13.6

Write two equilibrium conditions for the three tension members.



The equilibrium requirement is

$$P_{1y} + P_3 + P_{2y} = P \qquad 13.25$$

$$P_1 \cos\theta_1 + P_3 + P_2 \cos\theta_2 = P \qquad 13.26$$

Assuming the elongations are small compared to the member lengths, the angles θ_1 and θ_2 are unchanged. Then,

$$\frac{P_1 L_1}{A_1 E_1 \cos\theta_1} = \frac{P_3 L_3}{A_3 E_3} = \frac{P_2 L_2}{A_2 E_2 \cos\theta_2} \qquad 13.27$$

Equations 13.26 and 13.27 can be solved simultaneously to find P_1, P_2, and P_3. (It may be necessary to work with the x-components of the deflections in order to find a third equation.)

3 USING SUPERPOSITION WITH STATICALLY INDETERMINATE BEAMS

Continuous beams and propped cantilevers that are indeterminate to the first degree can often be solved by superposition.[2] This method requires finding the deflections with one or more of the supports removed, and then satisfying the given boundary conditions.

step 1: Remove the redundant supports to reduce the structure to a statically determinate condition.

step 2: Calculate the deflections at the previous locations of redundant supports. Use consistent sign conventions.

step 3: Apply each redundant support as a load, and find the deflections at the redundant support points as functions of the redundant support forces.

[2] Actually, this method can also be used with higher order indeterminate problems. However, the simultaneous equations that must be solved may make this method unattractive for manual solutions.

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